SOLUTION OF A LINEAR BOUNDARY VALUE PROBLEM OF RIEMANN FOR TWO FUNCTIONS AND ITS APPLICATION TO CERTAIN MIXED PROBLEMS IN THE PLANE THEORY OF ELASTICITY

(RESHENIE ODNOI LINEINOI KRAEVOI ZADACHI RIMANA DLIA DVUKH FUNKTSII I EE PRILOZHENIE K NEKOTORYM SMESHANNYM ZADACHAM PLOSKOI TEORII UPRUGOSTI)

PMM Vol.26, No.5, 1962, pp. 907-912

G.P. CHEREPANOV
(Moscow)

(Received April 12, 1962)

1. Boundary value problem. 1. Suppose that we are required to find two piece-wise analytic functions $\varphi_1(z)$ and $\varphi_2(z)$ of the complex variable z which satisfy on the contour L the equations

$$\varphi_1^+(t) = \alpha(t)\varphi_2^-(t) + f_1(t), \qquad \varphi_2^+(t) = \beta(t)\varphi_1^-(t) + f_2(t) \quad (1.1)$$

Here, $\alpha(t)$, $\beta(t)$, $f_1(t)$ and $f_2(t)$ are continuous almost everywhere on L and satisfy Holder's condition on the intervals of continuity ($\alpha(t)\beta(t) \neq 0$ on L).

In the general case, Riemann's boundary value problem for several functions can be reduced to a system of Fredholm equations. The most complete investigations of this problem are presented in a survey article of Gakhov [1], and in the monograph of Muskhelishvili [2]. In the work of Gakhov [3] there is obtained an effective closed solution for the case when the matrix coefficient of Riemann's problem is the product of two matrices whose elements are functions that are analytic in regions interior and exterior to L, respectively, except for a finite number of points at which these functions may have poles. In the article [1] there is given also a closed solution for the case when the coefficient in Riemann's problem is a functionally commutative matrix. The work [4] is devoted to some other cases of Riemann's problem for several functions with closed form solutions.

For a contour that divides the complex plane z into two regions, an interior one and an exterior one, the boundary value problem (1.1) is solved by means of a simple transformation of the function $\varphi_1^-(z)$ into the function $\varphi_2^-(z)$ and conversely.

2. Let us assume that the contour L consists of a certain number, n, of simple smooth arcs. The functions $\varphi_1(z)$ and $\varphi_2(z)$ will be assumed to be bounded at infinity.

By a canonic solution of the boundary value problem (1.1) for the given closed contour L we shall mean a pair of piece-wise analytic functions $X_1(z)$ and $X_2(z)$ satisfying on L the boundary conditions

$$X_1^+(t) = \alpha(t) X_2^-(t), \qquad X_2^+(t) = \beta(t) X_1^-(t) \tag{1.2}$$

and possessing the following properties:

1) The function $X_1(z)$ has ν_1 zeros at the points $z = c_i$ $(i = 1, \ldots, \nu_1)$, while the function $X_2(z)$ has ν_2 zeros at the points $z = d_i$ $(i = 1, \ldots, \nu_2)$. At every other point of the finite part of the plane each function is of zero order except, possibly, at the points of discontinuity of the coefficients $\alpha(t)$, and $\beta(t)$, and at the ends of the arcs L_k, the points $z = a_k$ and $z = b_k$ $(L = L_1 + \ldots + L_n)$.

The numbers ν_1, ν_2, c_i, d_i are such that the algebraic system consisting of the $n - 1)$ equations

$$\int_L \tau^k \ln\left[\frac{\alpha(\tau)}{\beta(\tau)} \prod_{i=1}^{\nu_1} (\tau - c_i)^{-2} \prod_{i=1}^{\nu_2} (\tau - d_i)^2\right] \frac{d\tau}{B_n^+(\tau)} = 0 \tag{1.3}$$

$$(k = 0, 1, \ldots, n-2)$$

is consistent. Here $B_n^+(\tau)$ denotes the limiting value on the left side of the cut of the function

$$B_n(z) = \prod_{i=1}^{n} (z - a_k)^{1/2} (z - b_k)^{1/2} \tag{1.4}$$

which is analytic in the exterior of the contour L.

2) At the ends $z = a_k$ and $z = b_k$ of the arcs, and at the discontinuity points of the coefficients $\alpha(t)$ and $\beta(t)$, the class of functions $X_j(z)$ coincides with the given class of functions $\varphi_j(z)$, $j = 1, 2$.

3) At infinity, the function $X_j(z)$ has the highest possible order. The boundary conditions (1.2) are equivalent to the following relations:

$$(X_1X_2)^+ = \alpha\beta\,(X_1X_2)^-, \qquad \left(\frac{X_1}{X_2}\right)^+ \left(\frac{X_1}{X_2}\right)^- = \frac{\alpha}{\beta} \tag{1.5}$$

Hence, for the product of the functions $X_1(z)$ and $X_2(z)$ we have the linear homogeneous problem of Riemann [2,5]; while the ratio of these functions is determined by a nonlinear boundary value problem of the Riemann type considered in the work [6]. We thus obtain the canonic solution of the boundary value problem (1.1) in the form

$$X_1(z) = \prod_{i=1}^{\nu_1}(z - c_i)\prod_{k=1}^{n}(z - b_k)^{-\varkappa_k}\exp\left\{\frac{1}{2}[\Gamma_0(z) + \Gamma(z)]\right\}$$
$$X_2(z) = \prod_{i=1}^{\nu_2}(z - d_i)\prod_{k=1}^{n}(z - b_k)^{-\varkappa_k}\exp\left\{\frac{1}{2}[\Gamma_0(z) - \Gamma(z)]\right\} \tag{1.6}$$

$$\Gamma_0(z) = \frac{1}{2\pi i}\int_L \ln\,[\alpha(\tau)\,\beta(\tau)]\frac{d\tau}{\tau - z} \tag{1.7}$$

$$\Gamma(z) = \frac{B_n(z)}{2\pi i}\int_L \ln\left[\frac{\alpha(\tau)}{\beta(\tau)}\prod_{i=1}^{\nu_1}(\tau - c_i)^{-2}\prod_{i=1}^{\nu_2}(\tau - d_i)^2\right]\frac{d\tau}{B_n^+(\tau)(\tau - z)}$$

Here the arguments of the functions $\alpha(t)$ and $\beta(t)$ on the arcs L_k $(k = 1, \ldots, n)$ are selected in such a way that during the continuous change of the argument from the point $z = a_k$ to the point $z = b_k$ the singularities of the functions $X_j(z)$ at the points $z = a_k$ and at the points of the discontinuity of the coefficients $\alpha(t)$ and $\beta(t)$ will be of the type as those of the given class $\varphi_j(z)$, while at the point $z = b_k$ the given class is obtained by a choice of an integer κ_k (entirely analogous to the procedure used for the linear boundary value problem for a single function [2,5]). If the algebraic system (1.3) has several solutions ν_1 and ν_2, then, for the fulfillment of the condition (3) of the definition, one has to select the smallest solution. Thus, the solution of the homogeneous problem (1.2) for the closed contour L has been reduced to the solution of the algebraic system (1.3).

Let us now establish the solvability of the algebraic system (1.3) for the unknown c_i and d_i when ν_1 and ν_2 are positive integers. Indeed, from the general theory it is known [2,1] that there exists a solution of the homogeneous boundary value problem (1.2) which is of finite order at infinity. The general solution of the homogeneous boundary value problem (1.2) in the class of functions analytic in the finite part of the plane is given by the formulas (1.6) and (1.7) for arbitrary ν_1, ν_2, c_i and d_i. The requirement of finiteness of the order of the solution of the homogeneous problem (1.2) at infinity guarantees the fulfillment of the conditions (1.3) for same ν_1, ν_2, c_i and d_i. We shall consider certain cases when the numbers ν_1, ν_2, c_i and d_i can be found by elementary means.

1) Case when $n = 1$. The canonic solution of (1.1) exists; hence one may set $\nu_1 = \nu_2 = 0$ in the formulas (1.6) and (1.7).

2) Case when the function $\alpha(t)/\beta(t)$ is the boundary value to the left of the cuts of L of a function analytic outside the contour L; this function has no zeros in the exterior of L, and is of zero order at infinity; furthermore $\alpha^+(z)/\beta^+(z) = C\alpha^-(z)/\beta^-(z)$, where C is some constant. In this case condition (1.3) is satisfied if one sets $\nu_1 = \nu_2 = 0$.

3. Making use of the canonic solution (1.6), (1.7), satisfying conditions (1.2), we can write the nonhomogeneous boundary value problem (1.1) in the form

$$\left(\frac{\varphi_1}{X_1}\right)^+ = \left(\frac{\varphi_2}{X_2}\right)^- + \frac{f_1}{X_1^+}, \qquad \left(\frac{\varphi_2}{X_2}\right)^+ = \left(\frac{\varphi_1}{X_1}\right)^- + \frac{f_2}{X_2^+} \tag{1.8}$$

The solution of the boundary value problem (1.8) can be expressed as:

$$\varphi_j(z) = X_j(z)\left\{\frac{1}{4\pi i}\int\limits_L\left[\frac{f_1(\tau)}{X_1^+(\tau)} + \frac{f_2(\tau)}{X_2^+(\tau)}\right]\frac{d\tau}{\tau - z} - \right. \qquad (j = 1, 2)$$

$$\left. - (-1)^j\frac{B_n(z)}{4\pi i}\int\limits_L\left[\frac{f_1(\tau)}{X_1^+(\tau)} - \frac{f_2(\tau)}{X_2^+(\tau)}\right]\frac{d\tau}{B_n^+(\tau)(\tau - z)} + P_\varkappa^{(j)}(z)\prod_{i=1}^{\nu_j}(z - z_i)^{-1}\right\} \tag{1.9}$$

$$(z_i = c_i \text{ when } j = 1 \text{ and } z_i = d_i \text{ when } j = 2)$$

$$P_\varkappa^{(j)}(z) = \gamma_\varkappa^{(j)}z^\varkappa + \ldots + \gamma_0^{(j)}, \qquad P_\varkappa^{(j)}(z) = 0 \quad \text{when } \varkappa < 0, \quad \gamma_i^{(j)} = \text{const}$$

When $\nu_j - \kappa + n - 1 \geqslant 1$, the following $(\nu_j - \kappa + n - 1)$ conditions have to be fulfilled in order that the nonhomogeneous problem (1.1) may have a solution

$$\int\limits_L\left[\frac{f_1(\tau)}{X_1^+(\tau)} - \frac{f_2(\tau)}{X_2^+(\tau)}\right]\frac{\tau^k d\tau}{B_n^+(\tau)} = 0 \qquad (k = 0, 1, \ldots, \nu_j - \varkappa + n - 2) \tag{1.10}$$

This condition (1.10) will hold only if $\nu_j - \kappa \leqslant 1$; as soon as $\nu_j - \kappa \geqslant 1$, other relations than those of (1.10) have to hold. These relations are obtained from the power expansion of the first two terms within the braces of Formula (1.9) in the neighborhood of infinity, and by equating to zero the coefficients of the positive powers of z in this expansion.

2. Basic mixed problem of the plane theory of elasticity for a plane with cuts along a straight line. 1. Let us assume that the region which is occupied by an elastic body is the entire plane cut along n line segments $L_k = a_k b_k$ $(k = 1, \ldots, n)$ of the real axis $x(L = L_1 + \ldots + L_n)$. On one part of the surface of the cuts M there are given stresses, on the other part, S, there are given displacements $(L = M + S)$.

Various particular cases of this problem have been considered by Muskhelishvili [7], and by Sherman [8,9].

The stresses and displacements in the plane problem of the theory of elasticity for bodies with cuts along the x-axis are described by means of potentials $\Phi(z)$ and $\Omega(z)$ according to Muskhelishvili in the following way:

$$\sigma_x + \sigma_y = 4\operatorname{Re}\Phi(z) \qquad (z = x + iy)$$

$$\sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (\bar{z} - z)\overline{\Phi'(z)}$$

$$2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \varkappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)} \tag{2.1}$$

Here σ_x, σ_y and τ_{xy} are the components of the stress tensor; u and v are the components of the displacement vector along the axes x and y; μ and ν are the translation modulus and Poisson's coefficient, respectively. Hereby, $\kappa = 3 + 4\nu$ for a planar deformation and $\kappa = (3 - \nu)/(1 + \nu)$ for a planar stress state.

For large values of z, the analytic functions $\Phi(z)$ and $\Omega(z)$ have the form [7]

$$\begin{aligned}\Phi(z) &= \frac{1}{4}(N_1 + N_2) - \frac{X + iY}{2\pi(1+\varkappa)}\frac{1}{z} + O(z^{-2}) \\ \Omega(z) &= \frac{1}{4}(N_1 + N_2) - \frac{1}{2}(N_1 - N_2)e^{2i\alpha} + \frac{\varkappa(X + iY)}{2\pi(1+\varkappa)}\frac{1}{z} + O(z^{-2})\end{aligned} \tag{2.2}$$

where (X, Y) is the principal vector of the external force applied to the edges of all the segments; N_1 and N_2 are the values of the principal stresses at infinity; α is the angle which the direction corresponding to N_1 makes with the x-axis.

It is easy to see from the expression (2.1) that the basic problems of the planar theory of elasticity for a plane with cuts can be reduced to a particular case of the problem considered in the preceding section when $\alpha(t)$ and $\beta(t)$ are piece-wise constant functions. Let us confine ourselves to the cases when the points on opposite sides of the cuts where the type of the boundary conditions change are symmetrically located with respect to the cut segments.

In this case the boundary value problem in the formulation (2.1) will take the form

$$\begin{aligned}\Phi^+(t) &= \alpha(t)\Omega^-(t) + f_1(t), \\ \Omega^+(t) &= \alpha^{-1}(t)\Phi^-(t) + f_2(t),\end{aligned} \qquad \alpha(t) = \begin{cases} -1 & (t \in M) \\ 1/\varkappa & (t \in S)\end{cases} \tag{2.3}$$

$$\begin{aligned} f_1(t) &= \begin{cases} \sigma_y^+ - i\tau_{xy}^+ & (t \in M) \\ 2\mu\varkappa^{-1}(\dot{u}^+ + i\dot{v}^+) & (t \in S)\end{cases} \\ f_2(t) &= \begin{cases} \sigma_y^- - i\tau_{xy}^- & (t \in M) \\ -2\mu(\dot{u}^- + i\dot{v}^-) & (t \in S)\end{cases}\end{aligned} \tag{2.4}$$

Here $\sigma_y{}^{\pm}$, $\tau_{xy}{}^{\pm}$, $\dot{u}^{\pm}$ and $\dot{v}^{\pm}$ are given stresses, and derivatives of the displacements on the edges of the cuts.

The solution of the boundary value problem (2.3) can be found by means of Formulas (1.6), (1.7), (1.9) and (1.3) in which the function $\alpha(t)$ is determined by Formula (2.3), while the function $\beta(t)$ is given by the equation

$$\beta(t) = \alpha^{-1}(t)\, e^{2\pi i n} \tag{2.5}$$

where n is an integer, which in general changes when the variable passes through a point where the boundary conditions change. The argument of the function $\alpha(t)$ on the segment L_k, and the integers n, and also the numbers κ_k are chosen in Formulas (1.6), (1.7), (1.9), (1.3) in such a way that the function $X_j(z)$, and, hence, also the functions $\Phi(z)$ and $\Omega(z)$ may have integrable infinite discontinuities at the ends $z = a_k$, $z = b_k$ of the segments and at the points of discontinuity of the coefficient $\alpha(t)$.

For the determination of the unknown coefficients of the polynomials $P_\kappa(z)$ in Formula (1.9) one may use [7] the following conditions: 1) the condition (2.2) at infinity; 2) the conditions of single-valuedness of the displacements; 3) the conditions expressing the fact that the displacements take on given values on the cuts.

Remark. Making use of the formulas of Muskhelishvili [7, Section 124], one can solve in a completely analogous way the basic mixed problem of the plane theory of elasticity for a plane with cuts along arcs of the circumference of a circle.

2. Let us consider a concrete example. Suppose that we are given the semi-infinite cut $(-\infty, +l)$. On one part $(-\infty, 0)$ of it there are known the displacements $u^{\pm} = 0$, $v^{\pm} = \pm h$ (h = const), while on the remaining part $(0, l)$, which is free of loading, we have $\sigma_y{}^{\pm} = \tau_{xy}{}^{\pm} = 0$. All stresses vanish at infinity. In this case, the functions $f_1(t)$ and $f_2(t)$ are zero, and the coefficient $\alpha(t)$ is given by

$$\alpha(t) = \begin{cases} -1 & (0, +l) \\ 1/\varkappa & (-\infty, 0) \end{cases} \tag{2.6}$$

For large values of z, the analytic functions $\Phi(z)$ and $\Omega(z)$ have the form

$$\Phi(z) = O(z^{-1}), \quad \Omega(z) = O(z^{-1}) \tag{2.7}$$

From Formulas (1.6), (1.7) and (1.9) we obtain

$$\Phi(z) = \frac{C}{\sqrt{z(z-l)}}\left[\frac{\sqrt{l}-\sqrt{l-z}}{\sqrt{l}+\sqrt{l-z}}\right]^{i\beta} \qquad (C = \mathrm{const}) \tag{2.8}$$

$$\Omega(z) = \frac{C}{\sqrt{z(z-l)}}\left[\frac{\sqrt{l}-\sqrt{l-z}}{\sqrt{l}+\sqrt{l-z}}\right]^{-i\beta} \qquad \left(\beta = \frac{\ln \varkappa}{2\pi}\right) \tag{2.9}$$

Here, if $z \to \infty$, we have

$$\sqrt{z(z-l)} = z + O(z^{-1}), \qquad \left(\frac{\sqrt{l}-\sqrt{l-z}}{\sqrt{l}+\sqrt{l-z}}\right)^{i\beta} = e^{-\pi\beta} + O(z^{-1}) \tag{2.10}$$

The function $\sqrt{(z-1)}$ is positive when $x > l$. By means of a formula of Muskhelishvili [7] the displacements may be written in the form

$$2\mu(u+iv) = \varkappa\varphi(z) - \omega(\bar{z}) - (z-\bar{z})\overline{\Phi(z)}$$

$$\varphi(z) = \int \Phi(z)\,dz, \quad \omega(z) = \int \Omega(z)\,dz \tag{2.11}$$

In accordance with the boundary conditions, the displacement vector will change by $2hi$ when its terminal point circumscribes the point at infinity.

From this, and from Formulas (2.8), (2.9) and (2.11), we obtain

$$C = \frac{\mu h}{\pi\sqrt{\varkappa}} \tag{2.12}$$

The considered case occurs, for example, if one splits an elastic body with a perfectly rigid wedge of thickness $2h$ when the friction coefficient between the wedge and the elastic body is very great (larger than 0.5, see [10]). Hereby, the length of the unstable crack l can be determined from the condition of Khristianovich on the finiteness of the stresses on the edge of the crack, and from two hypotheses of Barenblatt [11]

$$l = \frac{4\pi^2}{K^2}C^2 = \frac{E^2h^2}{K^2(1+\nu)^2(3-4\nu)} \tag{2.13}$$

Here E is Young's modulus; K is the modulus of cohesion [11].

For the comparison of lengths of cracks, we give the expression of the ratio of the length l of the crack in the considered case to the length l_0 of a crack obtained [11] by splitting of an infinitely brittle body with a perfectly rigid smooth wedge of thickness $2h$

$$\frac{l}{l_0} = \frac{4(1-\nu)^2}{3-4\nu}, \qquad l_0 = \frac{E^2h^2}{4(1-\nu^2)^2K^2} \qquad (h = \text{const}) \tag{2.14}$$

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Translated by H.P.T.